Consensus Equilibrium: A Framework for Model Integration

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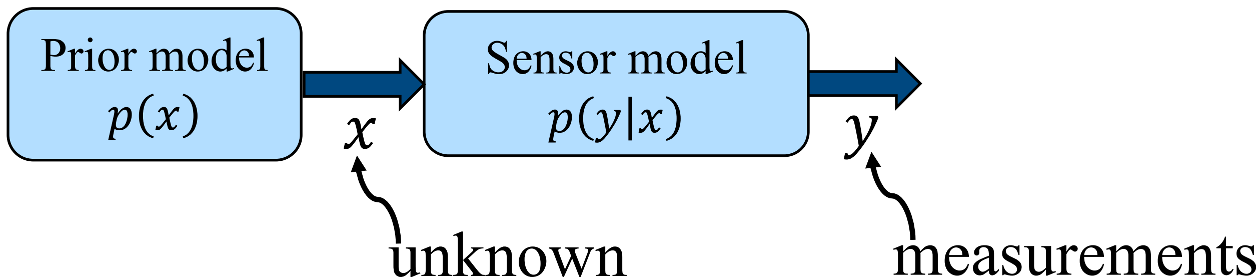
How can we integrate multiple heterogeneous models to yield a single coherent reconstruction?

- Our answer:
  - Plug-and-Play priors: An algorithm for regularized inversion
  - Consensus Equilibrium: A criteria for integration of models

MAP or Regularized Inversion

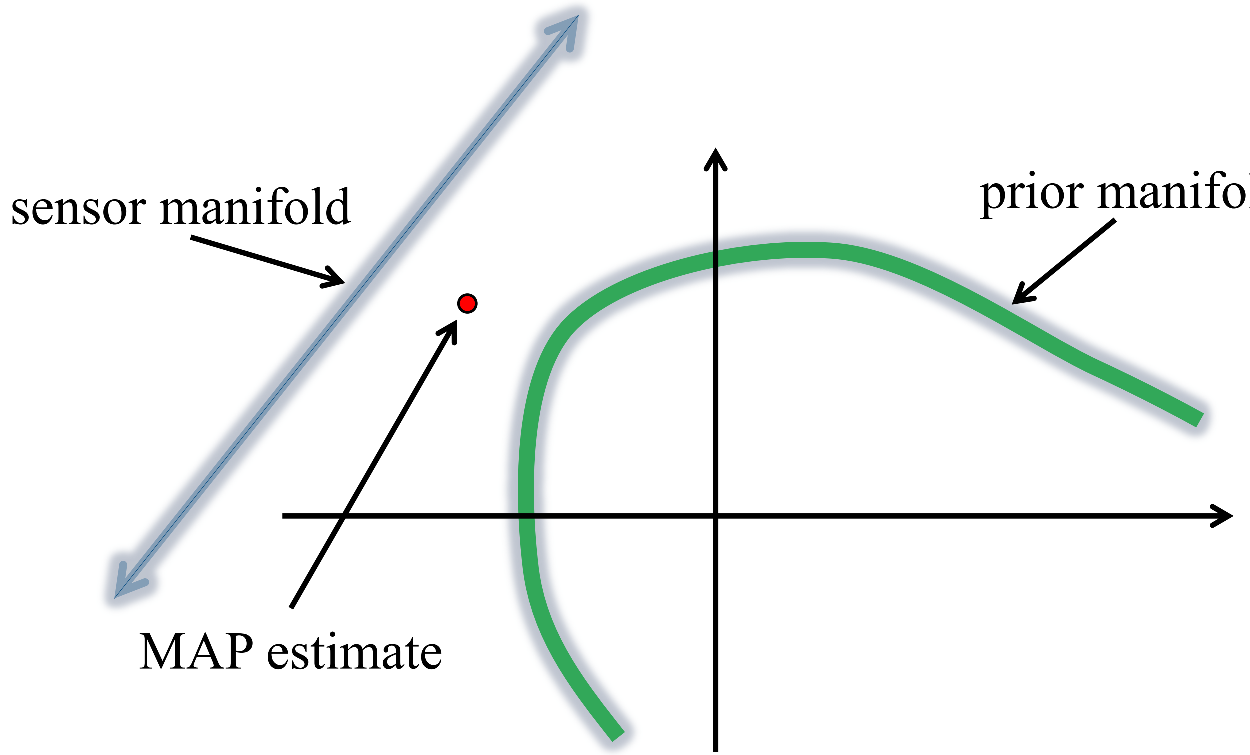


- Forward model: $f(x) = -\log p(y|x)$
- Prior model: $h(x) = -\log p(x)$

- MAP or regularized inverse

$$\hat{x} \leftarrow \arg\min_{x}\{f(x) + h(x)\}$$

“Thin Manifold” View of Multiple Models



- Sensor manifold – Based on physical sensor model
- Prior manifold – Based on empirical or assumed information
- MAP minimizes the sum of the costs

Proximal Maps

The **proximal map** is:

$$F(x) = \arg\min_{v} \left\{ f(v) + \frac{1}{2\sigma^2} \|v - x\|^2 \right\}$$

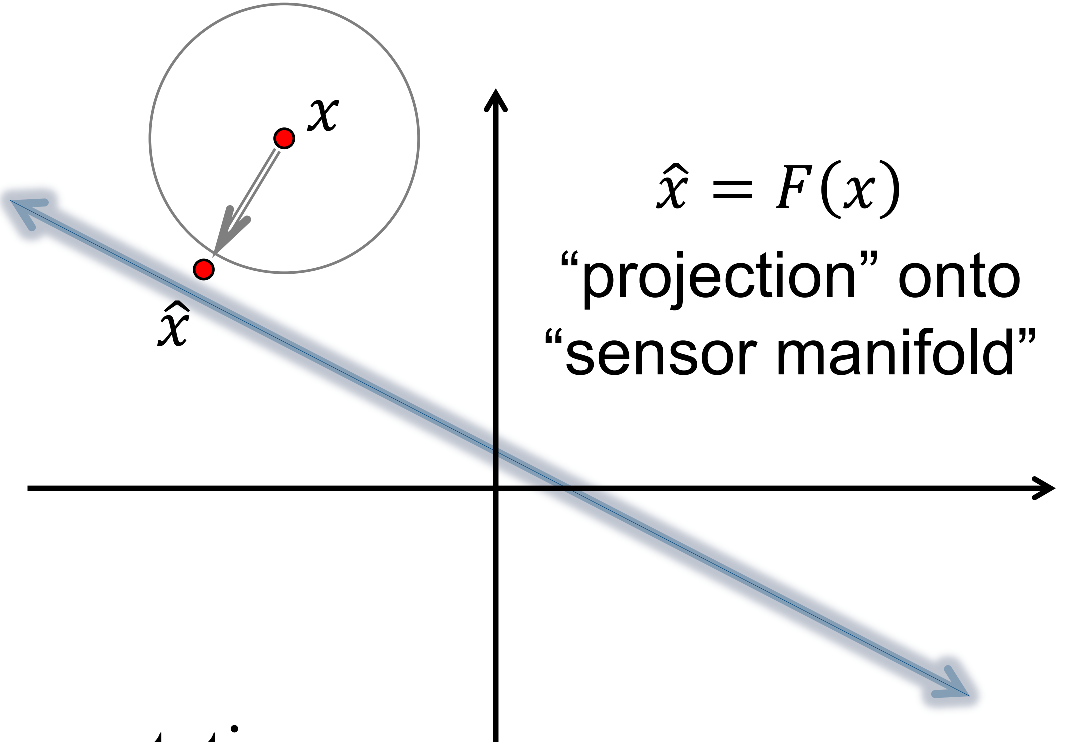
Intuition:

$$F(x) \approx x - \alpha \nabla f(x)$$

Sensor Proximal Map

sensor model such as $\frac{1}{2}\|y - Ax\|^2$

$$F(x) = \arg\min_v \left\{ f(v) + \frac{1}{2\sigma^2}\|v - x\|^2 \right\}$$

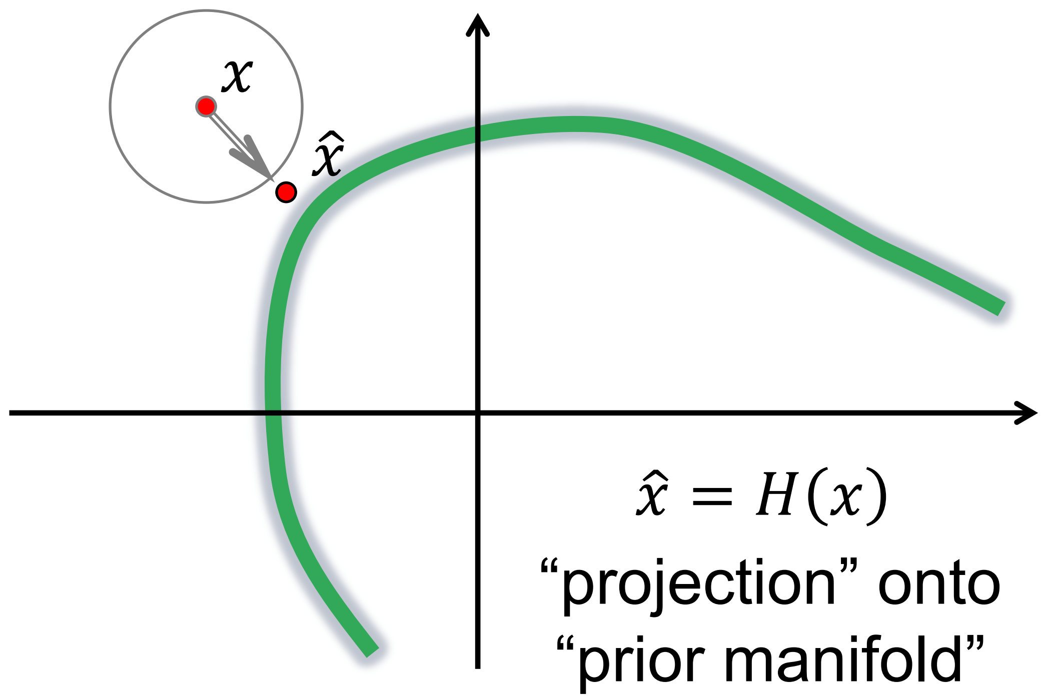


- Interpretations
  - "Projection" of x onto sensor manifold
  - MAP estimate with additive white Gaussian noise prior

Prior Proximal Map

prior model such as TV norm

$$H(x) = \arg\min_{v} \left\{ \frac{1}{2\sigma^2} \|v - x\|^2 + h(v) \right\}$$



- Interpretation
  - “Projection” of x onto prior manifold
  - Denoising operator for white additive Gaussian noise

Plug-and-Play Inversion
(ADMM algorithm with variable splitting)

Initialize $v, u = 0$

Repeat {

$x \leftarrow F(v - u)$ //Project onto sensor manifold

$v \leftarrow H(x + u)$ // **Denoise**

$u \leftarrow u + (x - v)$ // Augmented Lagrangian update

}

Big idea: Replace $H(x)$ with **any denoiser**

Big issue: This is **no longer minimization!!**

Inpainting with Many "Priors"

Ground Truth

Subsampled Image

Noise std. dev : 5% of max signal

K-SVD

RMSE : 14.11

BM3D

RMSE : 12.56

PLOW

RMSE : 14.54

TV

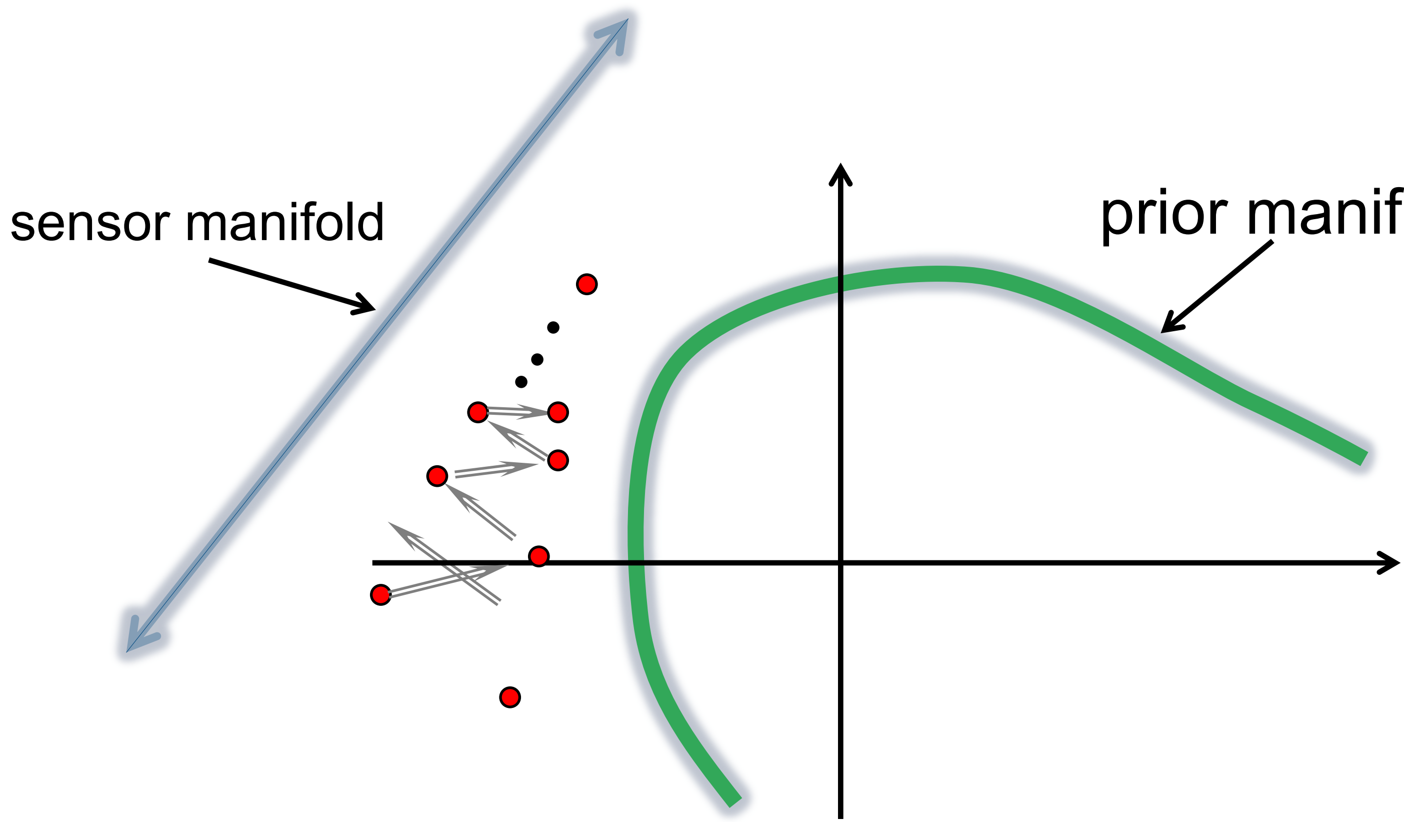
RMSE : 15.50

q-GGMRF

RMSE : 15.72

Plug-and-Play or Plug-and-Pray?

- Intuition: Alternating steps towards each manifold.



- Not minimizing anything!!!!!!
- It might converge…but to what?

Consensus Equilibrium

- What is Consensus Equilibrium?
  - A set of equilibrium conditions (like a PDE)
  - Generalization of regularized inversion
  - Generalization of Plug-and-Play
  - Allows integration of multiple Models

- Consensus Equilibrium is **not**
  - It is not an algorithm
  - It does not minimize a cost function

Analogy: Wave equation is a PDE, but it doesn't minimize energy

Consensus Equilibrium Equations

- P&P ADMM: Repeat

  step 1: $x \leftarrow F(v-u)$ // Forward step
  step 2: $v \leftarrow H(x+u)$ // Prior step
  step 3: $u \leftarrow u + (x-v)$ // Lagrangian

- If P&P convergences, then it must result in...

$$\left.\begin{aligned} x &= F(x-u) \\ x &= H(x+u) \end{aligned}\right\} \text{Consensus Equilibrium (CE) equations}$$

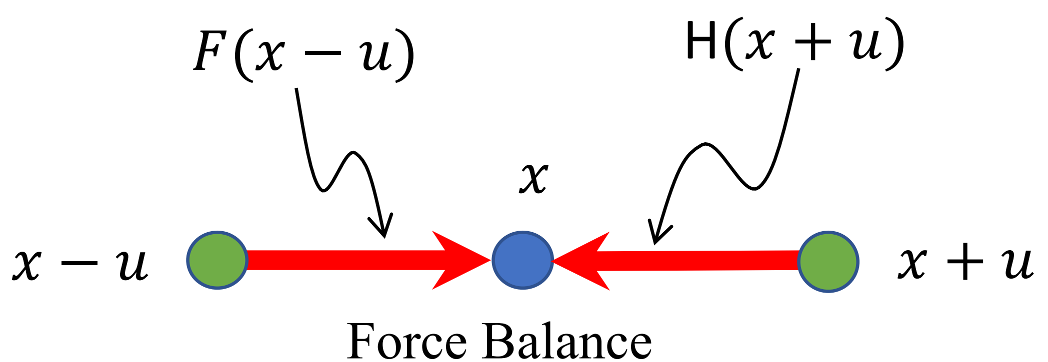
Consensus Equilibrium Equations

- All we really want is the solution (x, u) to…

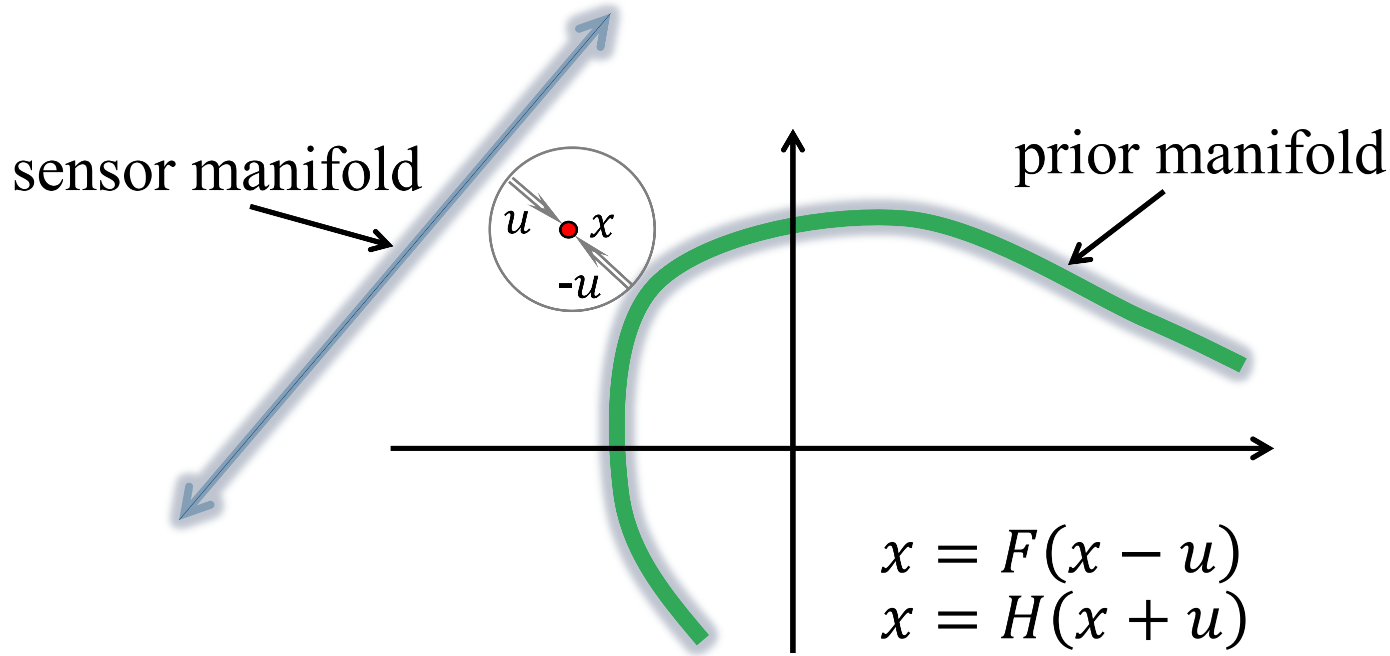
$$x = F(x - u) \qquad \text{(sensor agent)}$$

$$x = H(x + u) \qquad \text{(prior agent)}$$

  – Interpretation: u is the noise



Geometric Interpretation of Consensus Equilibrium



- Interpretation
  - u is the noise removed by the denoising operator H
  - Solution balances forces between F and H
  - Analog of department head job

Transformation of CE Equations

- By rotating coordinates, we get

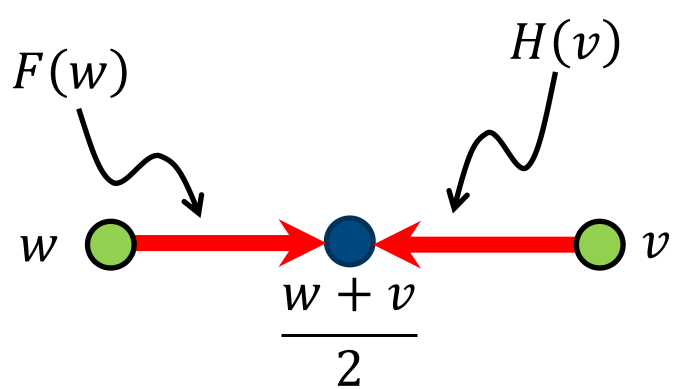
$$x = F(x-u)$$

$$x = H(x+u)$$

$$\Rightarrow$$

$$\frac{w+v}{2} = F(w)$$

$$\frac{w+v}{2} = H(v)$$



Multi-Agent of CE Equations

- Generalization for multiple models or agents

$$x = F_1(w_1)$$

$$x = F_2(w_2)$$

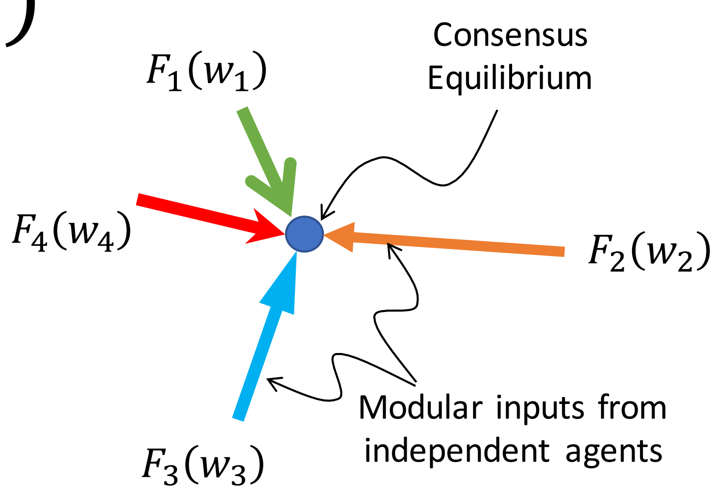
$$\vdots$$

$$x = F_N(w_N)$$

where

$$x = \frac{1}{N}\sum_{n=1}^{N} w_n$$

These functions are "agents" but not always proximal map



Solving the CE Equations

- Douglas-Rachford Algorithm

$$w^{k+1} = w^k + \rho\left(Tw^k - w^k\right)$$

where

$$T = (2F - I)(2H - I)$$

- Important Facts:
  - Converges to fixed point when T is non-expansive and $\rho \in (0,1)$.
  - **Exactly the ADMM algorithm when $\boldsymbol{\rho = 1/2}$.**
  - Generalization of ADMM when $\rho \neq 1/2$.

Multi-Agent Consensus Equilibrium

- Compact form: define the operators

$$F(w) = \begin{bmatrix} F_1(w_1) \\ \vdots \\ F_N(w_N) \end{bmatrix} \qquad \text{where } w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

and

$$G(w) = \begin{bmatrix} \bar{w} \\ \vdots \\ \bar{w} \end{bmatrix} \qquad \text{where } \bar{w} = \frac{1}{N}\sum_{i=1}^{N} w_i$$

- Then the consensus equilibrium equations are given by

$$Fw = Gw$$

Fixed Point Operator for Multi-Agent Problem

- Consensus equilibrium

$$Fw = Gw$$

Then it's easy to show that $(2G - I)^{-1} = (2G - I)$.

Using simple algebra, we have that

$$Fw^* = Gw^*$$
$$(2F - I)w^* = (2G - I)w^*$$
$$(2G - I)^{-1}(2F - I)w^* = w^*$$
$$(2G - I)(2F - I)w^* = w^*$$

So the CE solutions are exactly the fixed points of

$$\boxed{T = (2G - I)(2F - I)}$$

Solving the Multi-Agent CE Equations

- Can be solved using Douglas-Rachford Algorithm

$$w^{k+1} = w^k + \rho\left(Tw^k - w^k\right)$$

where

$$T = (2G - I)(2F - I)$$

- Important Facts:
  - Converges to fixed point when T is non-expansive and $\rho \in (0,1)$.
  - **Exactly the consensus ADMM algorithm when $\boldsymbol{\rho = 1/2}$.**
  - Generalization of consensus ADMM when $\rho \neq 1/2$.

P&P with Deep Learning (CNN) Prior Model

Integrate Multiple CNN Denoisers

- **Goal:** Denoise image

$$y = x + n$$

- **Problem:** We would like to use CNN, but don't know the true noise level.
- **Approach:** Use CE to integrate 5 different CNN denoisers each trained for a different noise level

Multiple CNN Denoisers

Noiseless | Noisy $\sigma_\eta = 40/255$ | DnCNN_{10}, 16.67dB | DnCNN_{15}, 17.53dB

DnCNN_{25}, 19.92dB | DnCNN_{35}, 26.44dB | DnCNN_{50}, 27.39dB | CE, 27.77dB

- True noise level 40/255.
- CE beats each individual: (10, 15, 25, 35, 50)/255

Multiple CNN Denoisers

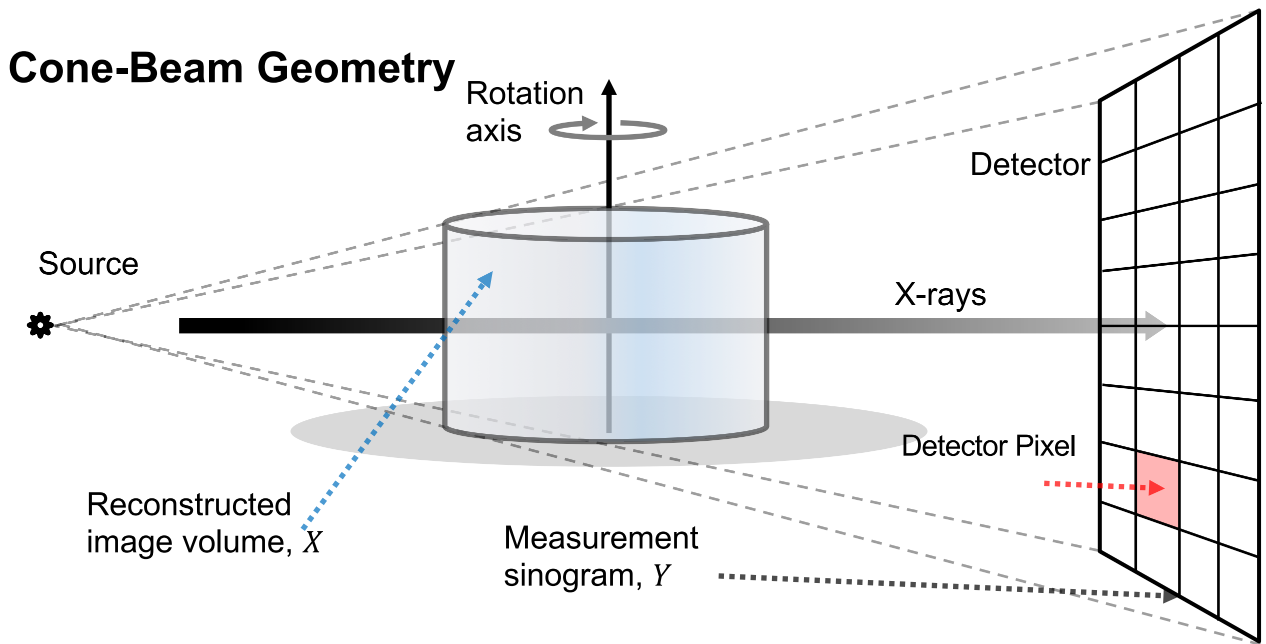
- CE outperforms mismatched CNNs, averaged denoisers without CE (baseline), and compares well with matched CNNs.

Image	DnCNN 10	DnCNN 15	DnCNN 25	DnCNN 35	DnCNN 50	Baseline	CE	Matched DnCNN
$\sigma = 20/255$								
Barbara512	23.99	28.02	30.49	28.11	25.71	29.80	**30.97**	31.02
Boat512	23.98	27.92	30.61	28.73	27.03	29.86	**31.08**	31.15
Cameraman256	24.12	28.04	30.20	28.52	27.20	29.88	**31.05**	31.07
Hill512	23.93	27.81	30.34	28.68	27.20	29.78	**30.88**	30.92
House256	24.03	28.70	33.70	32.32	30.69	31.38	**33.82**	33.97
Lena512	24.07	28.59	33.06	31.13	29.59	31.12	**33.35**	33.47
Man512	23.94	27.89	30.41	28.46	27.02	29.79	**31.00**	31.08
Peppers256	23.98	28.25	31.33	29.51	27.93	30.27	**31.79**	31.80

MBIR/P&P for Cone-Beam CT using Deep Learning Prior

Thilo Balke, Purdue University

Cone-Beam CT



- Nondestructive evaluation (NDE) of additively manufactured parts

Experimental Setup

Radiographs:

- GE Inspection Technologies v|tome|x C 450 HS with scatter|correct
- Dimensions: 700x800 pixels, 270 views
- Acceleration Voltage: 450 kV
- Current: 1.5 mA
- Exposure: 143 ms
- Scatter correction: GE-proprietary

Geometry:

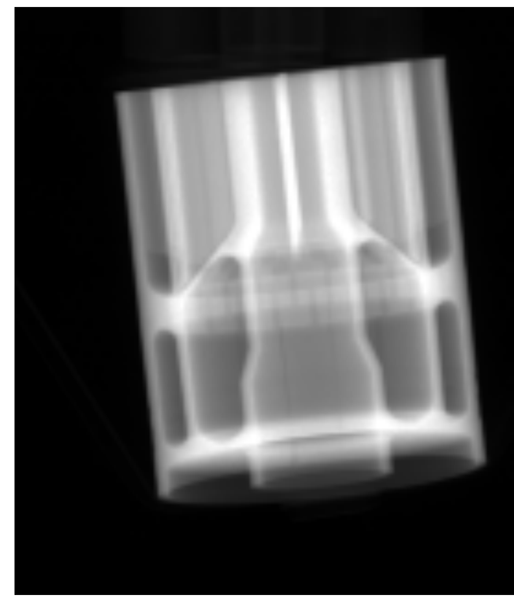
- Source-detector distance: 1160 mm
- Magnification: 1.5
- Detector pixels: $0.2 \times 0.2\ mm^2$

Object:

- 3D-printed CoCr part
- Diameter ≈ 60 mm
- Reconstruction resolution: (0.13 mm)

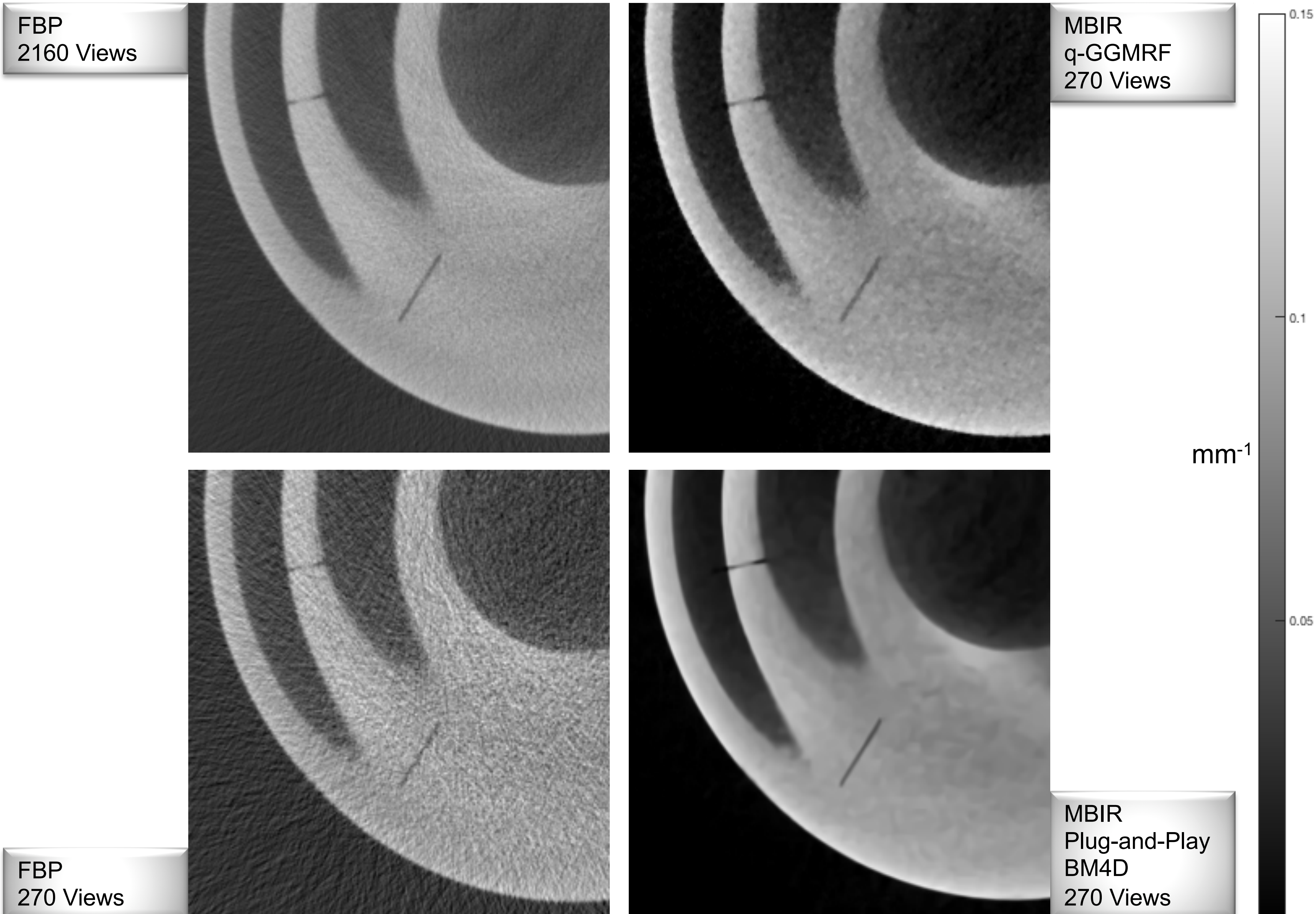
700x700x870

Sinogram Measurement



3D printed CoCr part

Reconstructions

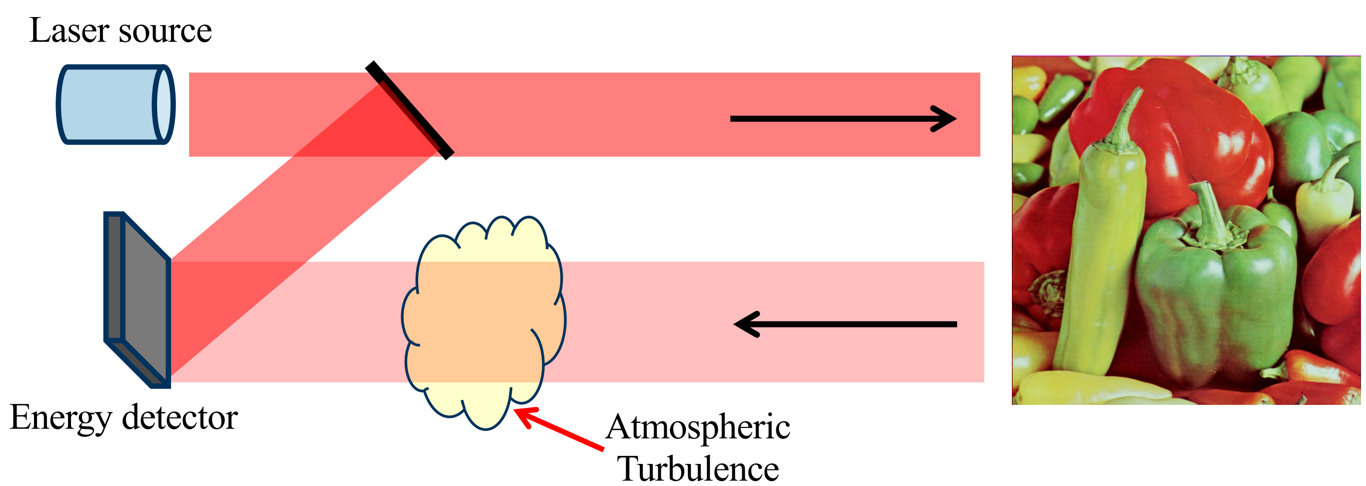


P&P for Coherent DH Imaging through Atmospheric Turbulence

Maj. Casey Pellizzari, United States Airforce Academy

Coherent Optical Imaging

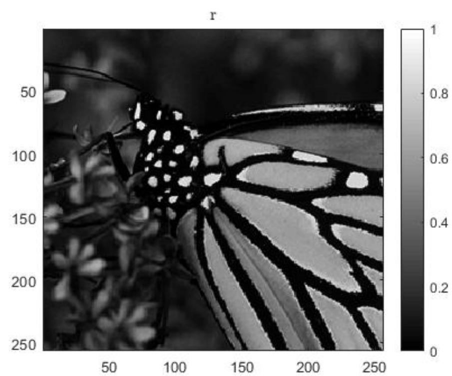
- Used P&P image prior in DH reconstruction



$$y = A_{\phi} g + w \text{ with } g \sim C(0, r)$$

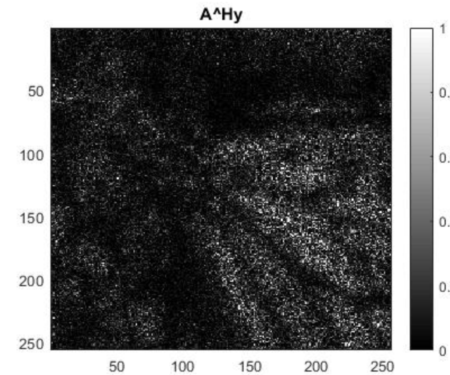
r – Unknown image
ϕ – Unknown phase distortion

CE Applied to Coherent Imaging (With Joint Phase Error Estimation)



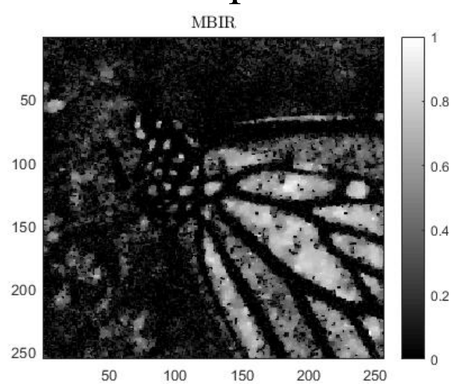
Actual Scene

Sensing Process →



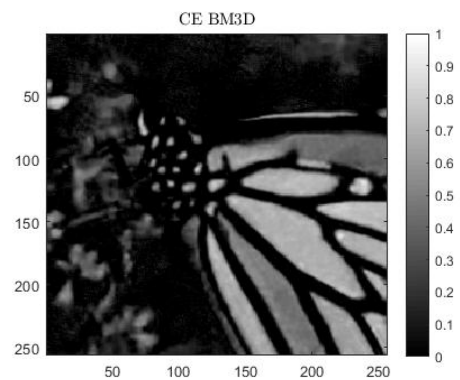
Blurry & Noisy Reconstruction

MRF prior



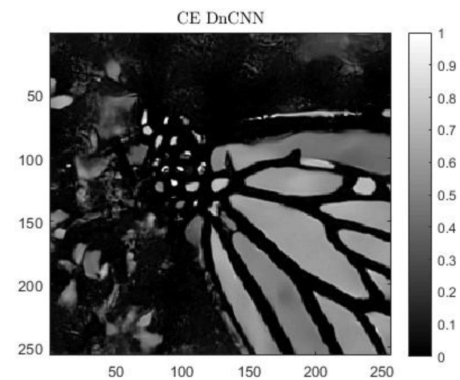
PSNR = 17.4
S = 0.53

P&P/CE: BM4D



PSNR = 18.6
S = 0.84

P&P/CE: DnCNN



PSNR = 22.0
S = 0.84

S = Strehl Ratio

Conclusions

- Consensus equilibrium viewpoint of P&P offers:
  - Flexible model integration without optimization
  - Integrates physical and machine learning models
  - Accommodates different numerical solvers
  - Makes nice images ☺